



Lemma 3.1

$(X, \Delta) \xrightarrow{\pi} U$ Proj contraction
 dit \mathbb{Q} -factorial affine, smooth, $d \leq k$, $0 \in U$

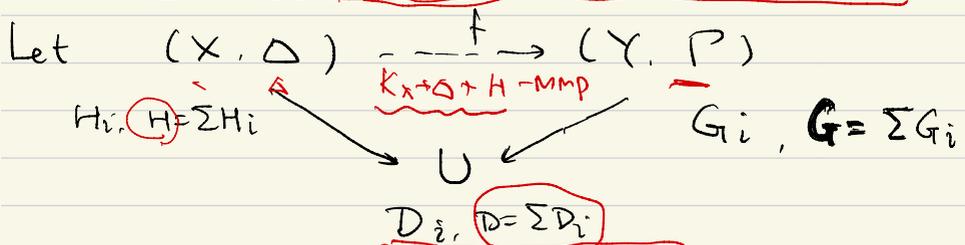
$H_i = \pi^* D_i$ $0 \in D_i, \quad i = 1, 2, \dots, k$

$H = \pi^* D$ $D = \sum D_i$

(1) $(X, \Delta + H)$ dit

(2) X_0 integral, $d_X X_0 = \dim X - d - U$
 $d_U V_0 = \dim V - \dim U$, where V is any non-canonical center V of (X, Δ)

(3) $B_-(K_{X_0} + \Delta_0)$ contains no non-canonical center of (X_0, Δ_0)



be a step of $K_X + \Delta - \text{MMP}/U$ which is birational,
 V is a non-canonical center of (X, Δ) . Then

f is iso at generic point of V and V_0

Let $W = f(V)$, then
 $V \xrightarrow{f|_V} W, V_0 \xrightarrow{f|_{V_0}} W_0$
 are birational

$(K_X + \Delta + H) - \text{MMP}$
 V is also a non-canonical center of $(X, \Delta + H)$
 V_0 is a non-canonical center of (X_0, Δ_0)

Further

$$\mathbb{P}_{V_0} \notin \mathbb{B}_-(K_{X_0} + \Delta_0)$$

(1) $(Y, P+G)$ is dlt ✓

(2) Y_0 is integral, $dc Y_0 = dc Y - dc U$

$dc W_0 = dc W - dc U$ for any non-canonical center W of (Y, P) ✓ V of (X, Δ)

(3) $\mathbb{B}_-(K_{Y_0} + P_0)$ contains no non-canonical center of (Y_0, P_0)

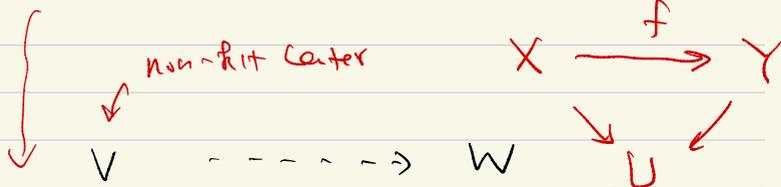
If V is a non-klt center, then

$V_0 \dashrightarrow W_0$ is a birational contraction.

If f is a MFS, then f_0 is not birational.

$$X_0 \dashrightarrow Y_0$$

pf:



$$(X, \Delta + H) \dashrightarrow^f (Y, P + G)$$

$$f|_{V_0}: V_0 \dashrightarrow W_0 \quad N := \text{Center of } P \text{ on } V_0$$

$$(K_{X_0} + \Delta_0)|_{V_0} = K_{V_0} + \Sigma_0$$

$$(K_{Y_0} + P_0)|_{W_0} = K_{W_0} + \Theta_0$$

$$a(P; V_0, \Sigma_0) < a(P; W_0, \Theta_0) \leq 1$$

N is a non-canonical center of (V_0, Σ_0)

if $\eta_2 \in \beta_-(\gamma_0, \beta_0)$, η_2 is also a non-cancel
Cent of (x_0, Δ_0)

$$\eta_2 \in \beta_-(K_{\gamma_0} + \beta_0)$$

$$\Rightarrow \eta_2 \in \beta_-(K_{\gamma_0} + \beta_0 + tA) \quad 0 < t < 1$$

$$\Rightarrow \eta_2 \in \beta_-(K_{x_0} + \Delta_0 + t(f^*A)) \quad \text{ample } A \text{ on } X_0$$

$$\eta_2 \notin \beta_-(f^*A)$$

$$f^*A \sim A' + E \Rightarrow \eta_2 \notin E$$

\uparrow
ample on X_0

$$\eta_2 \in \beta_-(K_{x_0} + \Delta_0 + tA' + tE) \quad \eta_2 \notin E$$

$$\Rightarrow \eta_2 \in \beta_-(K_{x_0} + \Delta_0 + tA')$$

$$\Rightarrow \eta_2 \in \beta_-(K_{x_0} + \Delta_0) \quad \text{Contradiction}$$

Lemma 3.2

$$\pi : (X, \Delta) \longrightarrow U \quad \text{proj contraction}$$

dit. \mathbb{Q} -fano alt. smooth, dlt
 $o \in U, \eta \in U$
 $H_i = \pi^* D_i$ $o \in D_i, D = \sum D_i$
 $H = \sum H_i$

- (1) $(X, \Delta + H)$ dlt
- (2) X_0 integral, $d_X X_0 = d_X X - d_X U$
 $d_X V_0 = d_X V - d_X U$ for any non-canonical center V of (X, Δ)
- (3) $B_-(K_{X_0} + \Delta_0)$ contains no non-canonical center of (X_0, Δ_0)

If (X_0, Δ_0) has a good minimal model, then we may run $(K_X + \Delta)$ -MMP $f: X \dashrightarrow Y$

until, $f_\eta: X_\eta \dashrightarrow Y_\eta$ is an (X_η, Δ_η) minimal model, and $f_0: X_0 \dashrightarrow Y_0$ is a semi-ample model of (X_0, Δ_0)
 $(X, \Delta + H)$ $(K_{Y_0} + \Delta_0)$

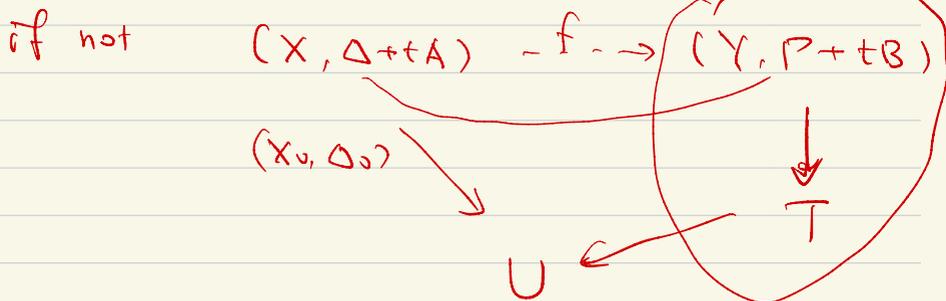
if D is a component of $L\Delta_0$ and $\phi: D \dashrightarrow E$ is the restriction of f to D , then

$\phi_0: D_0 \dashrightarrow E_0$ is a semi-ample model of (D_0, Σ_0)
 where $K_{D_0} + \Sigma_0 := (K_{X_0} + \Delta_0)|_{D_0}$

What's more, $B_-(K_X + \Delta)$ contains no non-canonical center of (X_0, Δ_0)

pf (X_0, Δ_0) admits a good minimal model

or $(K_X + \Delta)$ is pseudo-eff



$-(K_Y + P + tB)$ ample / T

$\Rightarrow Y_0$ is covered by $(K_{Y_0} + P_0 + tB_0)$ -negative curves

$K_{X_0} + \Delta_0 + tA_0 \rightsquigarrow K_{Y_0} + P_0 + tB_0$ is big

\downarrow

$\tilde{A} + E$

$\Rightarrow K_X + \Delta$ is pseudo-eff

$(X, \Delta + tA) \dashrightarrow (Y, P + tB) \dashrightarrow$

By Lemma 2.9.2: if $0 < t \ll 1$

(Y_0, P_0) is a semi-ample model of (X_0, Δ_0)

if we go on running nmp, the locus near Y_0 does not change

$\Rightarrow K_Y + P + tB$ is nef over generic point
for $0 < t < 1$

$$(X_\eta, \Delta_\eta) \dashrightarrow (Y_\eta, P_\eta)$$

$$(X, \Delta) \dashrightarrow (Y, P)$$

$$(X, \Delta + tA) \dashrightarrow (Y, P + tB)$$

$$B(K_X + \Delta + tA)$$

Theorem 4.1 (Berndtsson-Pauk)

Let $f: X \rightarrow \mathbb{P}^1$ be a proj contraction, and (X, Δ) is a log pair. If

(1) (X, Δ) log smooth over \mathbb{P}^1 and $L_{\Delta_0} = 0$

(2) the components of Δ do not intersect.

(3) $K_X + \Delta$ is pseudo-ef

(4) $B_-(K_X + \Delta)$ does not contain any component of Δ_0

then

$$H^0(m(K_X + \Delta)) \rightarrow H^0(m(K_{X_0} + \Delta_0))$$

is surj for any $m \in \mathbb{Z}^+$ s.t $m\Delta$ is integral.

In Theorem 4.2: $H^0(m(K_X + \mathbb{0})) \rightarrow H^0(m(K_{X_0} + \mathbb{0}_0))$
is surj

Theorem 4.2

Let $\pi: X \rightarrow U$ be a proj contractio, where U is smooth and (X, Δ) is log smooth over U s.t. $L\Delta_U = 0$. Then

$f_x \mathcal{O}_X(m(K_X + \Delta)) \rightarrow H^0(m(K_{X_u} + \Delta_u))$
is surj for any $m \in \mathbb{Z}^+$ and $u \in U$.

pf Assume U is affine, it suffices to prove

$$\left. \begin{array}{ccc} H^0(m(K_X + \Delta)) & \rightarrow & H^0(m(K_{X_u} + \Delta_u)) \\ H^0(m(K_X + \mathcal{O}_1)) & \rightarrow & H^0(m(K_{X_u} + \mathcal{O}_u)) \end{array} \right\} \text{is surj}$$

$$K_Y + P = f^*(K_X + \Delta) + F$$

when P, F have no common components

Just as we $K_{X_u} + \Delta_u$ is eff for $u \in U$

Least condition:

$B_-(K_X + \Delta)$ does not contain any component of Δ_u (non-canonical center of (X_u, Δ_u))

$$0 \cong \mathbb{H} \cong \Delta \quad (X_u, \mathcal{O}_u) \rightarrow (X, \mathcal{O})$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ u & \hookrightarrow & U \end{array} \quad |m(K_{X_u} + \Delta_u)|$$

$$m(K_{X_u} + \mathcal{O}_u)$$

any component of \mathcal{O}_0 is not contained in

$$B_-(K_{X_0} + \mathcal{O}_0)$$

After replacement of (X, Δ) by (X, \mathcal{O})

$B_-(K_{X_0} + \mathcal{O}_0)$ does not contain any non-Cartier
Center of (X_0, \mathcal{O}_0)

RR

$h^0(m(K_{X_u} + \Delta_u))$ is independent
of $u \in U$



