Lemma 3.1

$(X, \Delta) \xrightarrow{\pi} U$ prj contraction

dlt $\mathbb{Q}$-factorial — affine, smooth, $\dim k$, $0 \in U$

$H_i = \pi^* D_i$ — $0 \in D_i$, $i = 1, 2, \dots k$

$H = \pi^* D$ — $D = \sum D_i$

(1) $(X, \Delta + H)$ dlt.

(2) X_0 integral, $\dim X_0 = \dim X - \dim U$

$\dim V_0 = \dim V - \dim U$, where V is any non-canonical center V of (X, Δ)

(3) $B_-(K_{X_0} + \Delta_0)$ contains no non-canonical center of (X_0, Δ_0)

Let $(X, \Delta) \overset{f}{\dashrightarrow} (Y, \Gamma)$ — $K_X + \Delta + H$-MMP

H_i, $H = \sum H_i$ $\searrow$ U $\swarrow$ G_i, $G = \sum G_i$

D_i, $D = \sum D_i$

be a step of $K_X + \Delta$-MMP$/U$ which is birational, V is a non-canonical center of (X, Δ). Then

f is iso at generic point of V and V_0

Let $W = f(V)$, then $V \overset{f|_V}{\dashrightarrow} W$, $V_0 \overset{f|_{V_0}}{\dashrightarrow} W_0$ are birational

$(K_X + \Delta + H)$-MMP

V is also a non-canonical center of $(X, \Delta + H)$

V_0 is a non-canonical center of (X_0, Δ_0)

Further $\eta_{V_0} \notin B_-(K_{X_0}+\Delta_0)$

(1) $(Y, \Gamma+G)$ is dlt ✓

(2) Y_0 is integral. $\dim Y_0 = \dim Y - \dim U$

$\dim W_0 = \dim W - \dim U$ for any non-canonical center W of (Y, Γ) ✓ V of (X, Δ)

(3) $B_-(K_{Y_0}+\Gamma_0)$ contains no non-canonical center of (Y_0, Γ_0)

If V is a non-klt center, then $V_0 \dashrightarrow W_0$ is a birational contraction.

If f is a MFS, then f_0 is not birational.

$X_0 \dashrightarrow Y_0$

Pf:

$X \xrightarrow{f} Y$, $X \to U$, $Y \to U$

non-klt center: $X \to V$

$V \dashrightarrow W$

$(X, \Delta+H) \overset{f}{\dashrightarrow} (Y, \Gamma+G)$

$N :=$ Center of P on V_0

$f|_{V_0} : V_0 \dashrightarrow W_0$ P

$$(K_{X_0}+\Delta_0)|_{V_0} = K_{V_0}+\Sigma_0 \qquad (K_{Y_0}+\Gamma_0)|_{W_0} = K_{W_0}+\Theta_0$$

$a(P; V_0, \Sigma_0) < a(P; W_0, \Theta_0) \leq 1$

N is a non-canonical center of (V_0, Σ_0)

If Z, $(Y_0\ P_0)$, η_Z is also a non-canonical center of $(X_0.\ \Delta_0)$

$\eta_Z \in B_-(K_{Y_0}+P_0)$

$\Rightarrow \eta_Z \in B(K_{Y_0}+P_0+tA)$ $\quad 0<t\ll 1$ ample A on Y_0

$\Rightarrow \eta_Z \in B(K_{X_0}+\Delta_0+t(f^*A))$

$$\eta_Z \notin B(f^*A)$$

$f^*A \sim A' + E \geq 0 \quad \eta_Z \notin E$

A' ample on X_0

$$\eta_Z \in B(K_{X_0}+\Delta_0+tA'+tE) \qquad \eta_Z \notin E$$

$\Rightarrow \eta_Z \in B(K_{X_0}+\Delta_0+tA')$

$\Rightarrow \eta_Z \in B_-(K_{X_0}+\Delta_0)$ Contradiction.

Lemma 3.2

$\pi : (X, \Delta) \longrightarrow U$ proj contraction

dlt. $\mathbb{Q}$-factorial   affine. smooth, dim k

$o \in U$, $\eta \in U$

$H_i = \pi^* D_i$ $o \in D_i$, $D = \sum D_i$

$H = \sum H_i$

(1) $(X, \Delta + H)$ dlt

(2) X_o integral, $\dim X_o = \dim X - \dim U$

$\dim V_o = \dim V - \dim U$ for any non-canonical center V of (X, Δ)

(3) $B_-(K_{X_o} + \Delta_o)$ contains no non-canonical center of (X_o, Δ_o)

If (X_o, Δ_o) has a good minimal model then we may run $(K_X + \Delta)$-MMP $f : X \dashrightarrow Y$ until $f_\eta : X_\eta \dashrightarrow Y_\eta$ is an (X_η, Δ_η) minimal model, and $f_o : X_o \dashrightarrow Y_o$ is a semi-ample model of (X_o, Δ_o) $(K_{Y_o} + \Gamma_o)$

$(X, \Delta + H)$

if D is a component of $\lfloor \Delta \rfloor$ and $\phi : D \dashrightarrow E$ is the restriction of f to D, then

$\phi_o : D_o \dashrightarrow E_o$ is a semi-ample model of (D_o, Σ_o)

where $K_{D_o} + \Sigma_o := (K_{X_o} + \Delta_o)|_{D_o}$

What's more, $B_-(K_X + \Delta)$ contains no non-canonical center of (X_o, Δ_o)

Pf (X_0, Δ_0) admits a good minimal model

for $K_X + \Delta$ is pseudo eff

if not $(X, \Delta + tA) \overset{f}{\dashrightarrow} (Y, P + tB)$

$(X_0, \Delta_0) \searrow$

$$\begin{array}{ccc} & & (Y, P+tB) \\ & & \downarrow \\ U & \longleftarrow & T \end{array}$$

$-(K_Y + P + tB)$ ample $/T$

$\Rightarrow$ Y_0 is covered by $(K_{Y_0} + P_0 + tB_0)$-negative curves

$K_{X_0} + \Delta_0 + tA_0$ big $\rightsquigarrow$ $K_{Y_0} + P_0 + tB_0$ is big

$\sim \tilde{A} + E$

$\Longrightarrow$ $K_X + \Delta$ is pseudo eff

$(X, \Delta + tA) \dashrightarrow (Y, P + tB) \dashrightarrow$

By Lemma 2.9.2: if $0 < t \ll 1$

(Y_0, P_0) is a semiample model of (X_0, Δ_0)

If we go on running mmp, the locus near Y_0

does not change

$\Rightarrow$ $K_Y + P + tB$ is nef over generic point

for $0 < t \ll 1$

$(X_\eta, \Delta_\eta) \dashrightarrow (Y_\eta, P_\eta)$

$(X, \Delta) \dashrightarrow (Y, P)$

$(X, \Delta + tA) \dashrightarrow (Y, P + tB)$

$B(K_X + \Delta + tA)$

Theorem 4.1 (Berndtsson-Paun)

Let $f: X \to \mathbb{D}$ be a proj contraction, and (X, Δ) is a log pair. If

(1) (X, Δ) log smooth over $\mathbb{D}$ and $\lfloor \Delta \rfloor = 0$

(2) the components of Δ do not intersect.

(3) $K_X + \Delta$ is pseudo-eff

(4) $B_-(K_X + \Delta)$ does not contain any component of Δ_0

then

$$H^0(m(K_X+\Delta)) \to H^0(m(K_{X_0}+\Delta_0))$$

is surj for any $m \in \mathbb{Z}^+$ s.t. $m\Delta$ is integral.

In Theorem 4.2: $H^0(m(K_X+\Theta)) \to H^0(m(K_{X_0}+\Theta_0))$ is surj

Theorem 4.2

Let $\pi : X \to U$ be a proj contraction, where U is smooth and (X, Δ) is log smooth over U s.t $\lfloor \Delta \rfloor = 0$. Then

$$f_* \mathcal{O}_X(m(K_X+\Delta)) \longrightarrow H^0(m(K_{X_u}+\Delta_u))$$

is surj for any $m \in \mathbb{Z}^+$ and $u \in U$.

Pf   Assue U is affine, it suffices to prove

$$H^0(m(K_X+\Delta)) \longrightarrow H^0(m(K_{X_u}+\Delta_u)) \text{ is surj}$$

$$H^0(m(K_X+\Theta)) \longrightarrow H^0(m(K_{X_u}+\Theta_u))$$

$$K_Y+P = f^*(K_X+\Delta)+F$$

when P, F have no common components

Just assue $K_{X_u}+\Delta_u$ is eff for $u \in U$

Least condition:

$B_-(K_X+\Delta)$ does not contain any compont of Δ_u (non-canonical cents of (X_u, Δ_u))

$0 \leq \Theta \leq \Delta$

$$\begin{array}{ccc} (X_u, \Theta_u) & \to & (X, \Theta) \\ \downarrow & & \downarrow \\ u & \hookrightarrow & U \end{array}$$

$|m(K_{X_u}+\Delta_u)|$

$m(K_{X_u}+\Theta_u)$

any component of Θ_0 is not contained in

$B_-(K_{X_0}+\Theta_0)$

After replacement of (X, Δ) by (X, Θ)

> $B_-(K_{X_0}+\Theta_0)$ does not contain any non-canonical center of (X_0, Θ_0)

RK $h^0(m(K_{X_u}+\Delta_u))$ is independent of $u \in U$